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Uniformization and the Poincaré metric on the leaves of a foliation by curves

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Abstract. In this paper we prove that a holomorphic foliation by curves, on a complex compact manifold M, whose singularities are non degenerated and whose tangent line bundle admits a metric of negative curvature, satisfies the following properties: **(a):** All leaves are hyperbolic. **(b):** The Poincaré metric on the leaves is continuous. **(c):** The set of uniformizations of the leaves by the Poincaré disc $\mathbb{D}$ is normal. Moreover, if $(\alpha_n)_{n\geq 1}$ is a sequence of uniformizations which converges to a map $\alpha\colon \mathbb{D} \to M$, then either α is a constant map (a singularity), or α is an uniformization of some leaf. This result generalizes Theorem B of [LN], in which we prove the same facts for foliations of degree ≥ 2 on projective spaces.

Keywords: holomorphic foliations, Poincaré metric on the leaves, uniformization of the leaves.

1 Introduction

Let $\mathcal{F}$ be a holomorphic foliation by curves, with isolated singularities, in a complex compact manifold of dimension $n \geq 2$, say M. We will denote by $sing(\mathcal{F})$ the set of singularities of $\mathcal{F}$ and by $\mathcal{H}(\mathbb{D}, \mathcal{F})$ the set

$$\mathcal{H}(\mathbb{D}, \mathcal{F}) = \{\alpha\colon \mathbb{D} \to M\,;\ \alpha \text{ is a holomorphic and } \alpha(\mathbb{D}) \text{ is contained in some leaf of } \mathcal{F}\}$$

with the topology of uniform convergence on the compact parts of $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$. In this paper we will deal with the following questions:

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Question 1. When all leaves of $\mathcal{F}$ are uniformized by the Poincaré disk? If this is the case, we will say that the foliation is *hyperbolic* and we will denote by $\mathcal{U}(\mathcal{F})$ the set

$$\mathcal{U}(\mathcal{F}) = \{\alpha \in \mathcal{H}(\mathbb{D}, \mathcal{F});\ \alpha \text{ is an uniformization of some leaf of } \mathcal{F}\}.$$

Question 2. Let $\mathcal{F}$ be a hyperbolic foliation on M. When $\mathcal{U}(\mathcal{F})$ is normal? When the Poincaré metric on the leaves of $\mathcal{F}$ is continuous?

Let us clarify the last question. Fix a point $p \in M \setminus sing(\mathcal{F})$ and $(A, (x, y))$ be a foliated chart, where $p \in A$ and $x\colon A \to \mathbb{C}$, $y\colon A \to \mathbb{C}^{n-1}$ are such that $x(p) = 0$, $y(p) = 0$ and $\mathcal{F}|_A$ is defined by $dy = 0$. Since the leaves of $\mathcal{F}$ are hyperbolic and the Poincaré metric in a hyperbolic Riemann surface is unique, the Poincaré metric of the leaf passing through $(0, y)$ can be written as $\mu_P = f_A(x, y)|dx|^2$. Of course, the function f_A is real analytic with respect to the variable x, but it could be not continuous with respect to y. Let us observe that, if $(B, (u, v))$ is another foliated chart such that $A \cap B \neq \emptyset$, then $u = U(x, y)$ and $v = V(y)$ in $A \cap B$, so that, in this new coordinate system μ_P can be written as $f_B(u, v)|du|^2$, where $f_A(x, y) = f_B(U(x, y), V(y)) \cdot |U_x(x, y)|^2$. Therefore, the functions f_A and f_B have the same class. We say that the *Poincaré metric on the leaves of $\mathcal{F}$ is continuous*, if f_A, defined as above, is continuous for every foliated chart $(A, (x, y))$. In fact, it is known that the Poincaré metric on the leaves of $\mathcal{F}$ is continuous if, and only if, $\mathcal{U}(\mathcal{F})$ is normal and for any convergent sequence $(\alpha_n)_{n \geq 1}$ in $\mathcal{U}(\mathcal{F})$, where $\alpha_n \to \alpha$ and $\alpha(0) \notin sing(\mathcal{F})$, then $\alpha \in \mathcal{U}(\mathcal{F})$ (cf. [V], [C] and [LN]).

In this paper we intend to generalize some of the results of [LN], which were proved for singular holomorphic foliations on projective spaces. In order to state our main result we recall the concept of tangent bundle associated to a holomorphic foliation. A foliation $\mathcal{F}$ on a complex manifold M can be defined by an open covering $(U_\alpha)_{\alpha \in A}$, a colection of holomorphic vector fields $(X_\alpha)_{\alpha \in A}$ and a multiplicative cocycle $(f_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ such that (cf. [Br]):

(i) X_α is a holomorphic vector field on U_α.

(ii) If $U_\alpha \cap U_\beta \neq \emptyset$ then $f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $X_\beta = f_{\alpha\beta} \cdot X_\alpha$ on $U_\alpha \cap U_\beta$.

The *tangent bundle* of the foliation $\mathcal{F}$ is the holomorphic line bundle associated to the cocycle $(f_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$. We will denote this line bundle by $T_\mathcal{F}$ and its dual by $T^*_\mathcal{F}$. Now we can state our main result.

Theorem 1. *Let $\mathcal{F}$ be a foliation by curves on a compact complex manifold M. Suppose that $T_{\mathcal{F}}$ has a metric of negative curvature and that all singularities of $\mathcal{F}$ are non degenerated. Then:*

(a) *All leaves of $\mathcal{F}$ are hyperbolic.*

(b) *$\mathcal{U}(\mathcal{F})$ is normal. Moreover, $\overline{\mathcal{U}(\mathcal{F})} = \mathcal{U}(\mathcal{F}) \cup sing(\mathcal{F})$, that is, if $(\alpha_n)_{n\geq 1}$ is a convergent sequence in $\mathcal{U}(\mathcal{F})$, say $\alpha_n \to \alpha$, then, either $\alpha \in \mathcal{U}(\mathcal{F})$, or $\alpha \equiv c$, is a constant map, where $c \in sing(\mathcal{F})$.*

(c) *The Poincaré metric on the leaves of $\mathcal{F}$ is continuous.*

As a consequence, we have the following result, which includes Theorem B of [LN]:

Corollary 1. *Let $\mathcal{F}$ be a foliation by curves on a compact complex manifold M. Suppose that all singularities of $\mathcal{F}$ are non degenerated and $T^*_{\mathcal{F}}$ is ample. Then $\mathcal{F}$ satisfies* **(a)**, **(b)** *and* **(c)** *of Theorem 1. In particular, if $\mathcal{F}$ is a foliation on $\mathbb{CP}(n)$ of degree $d \geq 2$ with non degenerated singularities, then $\mathcal{F}$ satisfies* **(a)**, **(b)** *and* **(c)** *of Theorem 1.*

Proof. It is well known that if L is a ample line bundle on M, then L^* has a metric of negative curvature. This implies the first assertion. On the other hand, if $\mathcal{F}$ is a foliation on $\mathbb{CP}(n)$ of degree d, then $T^*_{\mathcal{F}} = (d-1)H$, where H denotes the divisor of a hyperplane. It follows that, if $d \geq 2$ then $T^*_{\mathcal{F}}$ is ample. This implies the last assertion. □

The following result is a consequence of Corollary 1 and of the Nakai-Moishezon criterion (cf. [F] pg. 18).

Corollary 2. *Let $\mathcal{F}$ be a foliation by curves on a compact complex surface M. Suppose that all singularities of $\mathcal{F}$ are non degenerated and that $(T^*_{\mathcal{F}})^2 > 0$ and $T^*_{\mathcal{F}} \cdot C > 0$ for all irreducible curve C on M. Then $\mathcal{F}$ satisfies* **(a)**, **(b)** *and* **(c)** *of Theorem 1.*

Example. We can apply Corollary 2 in the following case: Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{CP}(2)$ with $sing(\mathcal{F}) = \{p_1, ..., p_k, p_{k+1}, ..., p_N\}$, where $p_{k+1}, ..., p_N$ are non degenerated and $p_1, ..., p_k$ are degenerated singularities of $\mathcal{F}$. Suppose that:

(a) For each $j = 1, ..., k$, the foliation $\mathcal{F}$ is reduced (in the sense of Seidemberg [Se]) with just one blowing-up at point p_j. Denote by M the manifold obtained from $\mathbb{CP}(2)$ by blowing-up at the points $p_1, ..., p_k$, by $\pi : M \to \mathbb{CP}(2)$ the blowing-up map and by $\mathcal{G}$ the strict transform of $\mathcal{F}$ by π. Let $E_j = \pi^{-1}(p_j)$.

(b) All singularities of $\mathcal{G}$ are non degenerated.

(c) For each $j = 1, ..., k$, let m_j be the generalized multiplicity of $\mathcal{F}$ at p_j (cf. [Br]). Assume that:

(c.1) For any irreducible curve C on $\mathbb{CP}(2)$ we have

$$\sum_j \ell_j(m_j - 1) < \ell(d - 1)$$

where $\ell = dg(C)$ and $\ell_j = mult(C, p_j)$.

(c.2) $\sum_j (m_j - 1)^2 < (d - 1)^2$.

Then $\mathcal{G}$ satisfies **(a)**, **(b)** and **(c)** of Theorem 1.

The generalized multiplicity of $\mathcal{F}$ at p_j is defined as follows: let ω be a holomorphic 1-form which represents $\mathcal{F}$ in a neighborhood of p_j. Since p_j is a singularity of ω, the form $\pi^*(\omega)$ is identically zero along E_j. Then, m_j is order of E_j in the divisor of zeroes of $\pi^*(\omega)$.

Proof. We know that $T^*_{\mathcal{F}} = (d-1)H$, where H denotes the divisor of a hyperplane on $\mathbb{CP}(2)$. It follows from [Br] that $T^*_{\mathcal{G}} = (d-1)\tilde{H} - \sum_j (m_j - 1)E_j$, where $\tilde{H} = \pi^*(H)$. Now, as the reader can check, condition **(c.2)** implies that $(T^*_{\mathcal{G}})^2 > 0$ and condition **(c.1)** that $T^*_{\mathcal{G}} \cdot C > 0$, for any irreducible curve C on M. □

Remark 1. Concerning the hyperbolicity of the leaves of a foliation by curves, with degenerated singularities, the following result is known (cf. [G]):

Theorem. *Let $\mathcal{F}$ be a foliation on a nonsingular projective manifold defined by a meromorphic vector field with an ℓ-ample divisor, $\ell > 0$. Suppose that either $\mathcal{F}$ has no singularities or that it has isolated singularities, say $p_1, ..., p_s$, and that*

$$\sum_{j=1}^{s} (\mathcal{L}(\mathcal{F}, p_j) - 1) < \ell$$

where $\mathcal{L}(\mathcal{F}, p_j)$ is the Lojasiewicz exponent of $\mathcal{F}$ at p_j. Then all leaves of $\mathcal{F}$ are hyperbolic.

According to [G], a divisor D on M is *ℓ-ample* if there exists an embedding of M on a projective space, such that D is equivalent to $-\ell \cdot H$, where H is the hyperplane section of M. Therefore, the hypothesis of the Theorem, corresponds to the fact that the foliation $\mathcal{F}$ can be defined by a meromorphic vector field X on M such that its divisor of zeroes is empty and its divisor of poles is $\ell \cdot H$. The Lojasiewicz exponent of a holomorphic vector field X with an isolated singularity at $p \in \mathbb{C}^n$, is defined by

$$\mathcal{L}(X, p) = \min\{k > 0;\ |X(z)| \geq C|z - p|^k,\ C > 0, \text{ for any } z \text{ in a neighborhood of } p\}.$$

If $Y = f \cdot X$, where f is holomorphic and $f(p) \neq 0$, then $\mathcal{L}(X, p) = \mathcal{L}(Y, p)$. Therefore, $\mathcal{L}(X, p)$ depends only on the germ at p of the foliation generated by X. The Lojasiewicz exponent of $\mathcal{F}$ at an isolated singularity p is defined as

$$\mathcal{L}(\mathcal{F}, p) = \mathcal{L}(X, p),$$

where X is any holomorphic vector field defining $\mathcal{F}$ in a neighborhood of p. This result answers Question 1 in this case, but not Question 2, that is, it is not known if the foliation $\mathcal{F}$ satisfies properties **(b)** and **(c)** of Theorem 1. The proof of the above result in [G] is done by contructing a C^2 hermitian metric in $M \setminus sing(\mathcal{F})$ which induces strictly negative gaussian curvature in the leaves of $\mathcal{F}$. On the other hand, in the proof of Theorem 1 (and also of Theorem B of [LN]) we construct a continuous hermitian metric in $M \setminus sing(\mathcal{F})$ which is *complete* and induces an *ultrahyperbolic* metric (in the sense of Ahlfors [Ah-1] and [Ah-2]) on the leaves of $\mathcal{F}$. The completness of the metric implies that $\mathcal{F}$ satisfies **(b)** and **(c)** of Theorem 1. The following problem is natural:

Problem. Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities, on a complex compact manifold. Suppose that all leaves of $\mathcal{F}$ are hyperbolic. Is the Poincaré metric on the leaves of $\mathcal{F}$ continuous? Is $\mathcal{U}(\mathcal{F})$ normal and $\overline{\mathcal{U}(\mathcal{F})} = \mathcal{U}(\mathcal{F}) \cup sing(\mathcal{F})$?

We would like to observe that the answer is known to be positive only in two cases: the case of Theorem 1 and in the case in which the singular set of $\mathcal{F}$ is empty (cf. [C]).

2 Proof of Theorem 1

In the proof we will use the concept of $\mathcal{F}$-ultrahyperbolic metric and Ahlfor's lemma. In 1938 Ahlfors introduced the concept of *ultrahyperbolic* conformal metric in a Riemann surface (cf. [Ah-1] and [Ah-2]). A *conformal pseudo-riemannian metric* in a Riemann surface S, is a quadratic form g in S, which can be written in a holomorphic coordinate system (z, U) as $g_z = f(z)|dz|^2$, where f is a continuous function on the open set $z(U) \subset \mathbb{C}$, such that $f \geq 0$ and the set $f^{-1}(0)$ is discrete. In this case, we define the lengh of a C^1 curve $\gamma\colon [a, b] \to S$ with respect to g, by

$$\ell_g(\gamma) = \int_\gamma \sqrt{g} = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t))}dt\,. \tag{1}$$

It is not difficult to see that if $\gamma(a) \neq \gamma(b)$ then $\ell_g(\gamma) > 0$. Therefore we can define a distance in S by

$$d_g(p, q) = \inf\{\ell_g(\gamma);\ \gamma \text{ is a } C^1 \text{ curve joining } p \text{ and } q\}. \tag{2}$$

Definition 1. Let g be a conformal pseudo-riemannian metric in S. We say that g is *ultrahyperbolic of curvature bounded by* $a < 0$, if for every $p \in U$ with $g_p \neq 0$, there exist a holomorphic coordinate system (z, U) around p with $z(p) = 0$, $g|_U = f(z)|dz|^2$, $f > 0$ in $z(U) = V$,and a positive C^2 function h in V, such that $h(0) = f(0)$, $h \leq f$ in V and the gaussian curvature of the metric $h(z)|dz|^2$, say k, satisfies $k \leq a < 0$ in V. We say that g is *ultrahyperbolic*, if it is ultrahyperbolic of curvature bounded by $a < 0$, for some $a < 0$.

Remark 2. We would like to observe that Ahlfors' definition is more general, in the sense that he demands that f is just upper semicontinuous. However, in this paper all metrics that will appear will be continuous.

Remark 3. It is not difficult to see that this concept is well defined and invariant by biholomorphisms. Moreover, if S_1 and S_2 are two Riemann surfaces, $F\colon S_1 \to S_2$ is a holomorphic non constant map and g is a conformal pseudo-riemannian metric in S_2, ultrahyperbolic of curvature bounded by $a < 0$, then $F^*(g)$ is also.

The following result was proved by Ahlfors:

Theorem (Ahlfors Lemma). *Let S be a Riemann surface and suppose that there is a conformal pseudo-riemannian metric g in S which is ultrahyperbolic of curvature bounded by $-a^2 < 0$. Then S is hyperbolic and $g \leq \frac{1}{a^2}\mathbb{P}_S$, where $\mathbb{P}_S$ is the Poincaré metric of S. In particular, we have*

$$d_g(p, q) \leq \frac{1}{a} d_{\mathbb{P}}(p, q)\ ,\ \ \forall\, p, q \in S\,,$$

where $d_{\mathbb{P}}$ is the Poincaré distance in S.

Let us state anoother result that will be used.

Proposition 1. *Let S be a Riemann surface and $g_1, \ldots, g_k$ be ultrahyperbolic on S of curvature bounded by $-a_1 < 0, \ldots, -a_k < 0$, respectively. Define $g_M = \max\{g_1, \ldots, g_k\}$. Then g_M is ultrahyperbolic of curvature bounded by $-a_M < 0$, where $a_M = \min\{a_1, \ldots, a_k\}$ respectively.*

Proof. We observe that if $g_j = f_j(z)|dz|^2$, locally, then $\max\{g_1, \ldots, g_k\} = \max\{f_1, \ldots, f_k\}|dz|^2$. Therefore, the proof follows directly from Definition 1.□

We now introduce the notion of $\mathcal{F}$-*ultrahyperbolic* metric for a foliation $\mathcal{F}$.

Definition 2. Let M be a complex manifold of dimension ≥ 2 and $\mathcal{F}$ be a singular holomorphic foliation on M. We say that a continuous hermitian form H on M is a $\mathcal{F}$-*pseudo-metric*, if for any leaf $L \subset M \setminus sing(\mathcal{F})$ of $\mathcal{F}$, the quadratic form h_L defined on L by the restriction $H|_L$, is a conformal pseudo-riemannian metric on L. We say that H is $\mathcal{F}$-*ultrahyperbolic of curvature bounded by* $a < 0$, if for any leaf L of $\mathcal{F}$, h_L is ultrahyperbolic of curvature bounded by $a < 0$. We say that H is $\mathcal{F}$-*ultrahyperbolic*, if it is $\mathcal{F}$-ultrahyperbolic of curvature bounded by $a < 0$ for some $a < 0$. We say that H is complete, if the pseudo-distance d_H, defined on $M \setminus sing(\mathcal{F})$ by (1) and (2), is complete.

We now prove a foliated version of Ahlfors' lemma.

Proposition 2. *Let $\mathcal{F}$ be a singular holomorphic foliation on a complex compact manifold M. Suppose that there exists a continuous $\mathcal{F}$-ultrahyperbolic hermitian metric μ on $M \setminus sing(\mathcal{F})$. Then the following properties are true:*

(a) *All leaves of $\mathcal{F}$ are hyperbolic. Moreover, if there exists a metric δ on M such that $\delta \leq d_\mu$, where d_μ is the pseudo-distance induced by μ on $M \setminus sing(\mathcal{F})$, then $\mathcal{U}(\mathcal{F})$ is normal.*

(b) *Suppose that μ is complete in $M \setminus sing(\mathcal{F})$. Then the Poincaré metric on the leaves of $\mathcal{F}$ is continuous. Moreover, if $(\alpha_n)_{n\geq 1}$ is a convergent sequence in $\mathcal{U}(\mathcal{F})$, say $\alpha_n \to \alpha$ and $\alpha_n(0) \to p \in M$, then*

(b.1) *If $p \in sing(\mathcal{F})$, then $\alpha \equiv p$, is a constant.*

(b.2) *If $p \notin sing(\mathcal{F})$, then $\alpha \in \mathcal{U}(\mathcal{F})$.*

Proof. The first part of assertion **(a)** is immediate from Ahlfors' Lemma. On the other hand, the proof of assertion **(b)** is similar to the proof of Theorem A in [LN]. Let us prove the second part of assertion **(a)**. Suppose that μ is $\mathcal{F}$-ultrahyperbolic of curvature bounded by $-a^2 < 0$ and let δ be as in the hypothesis. Fix $\alpha \in \mathcal{U}$. It follows from Ahlfors' Lemma that

$$\delta(\alpha(z_2), \alpha(z_1)) \leq d_\mu(\alpha(z_2), \alpha(z_1)) \leq d_L(\alpha(z_2), \alpha(z_1)) \leq \frac{1}{a} d_P(z_2, z_1) ,$$

where d_L is the distance induced by d_μ on $L = \alpha(\mathbb{D})$ and d_P is the Poincaré distance on $\mathbb{D}$. In particular $\mathcal{U}(\mathcal{F})$ is equicontinuous with respect to δ and d_P. Now, fix $0 < r < 1$. Since M and $D_r := \{z \in \mathbb{D};\ |z| \leq r\}$ are compact, it follows from Arzela-Ascoli Theorem that the set

$$\mathcal{U}_r := \{\alpha|_{D_r};\ \alpha \in \mathcal{U}(\mathcal{F})\} \subset C^0(D_r, M)$$

is paracompact in the topology of uniform convergence in $C^0(D_r, M)$, if we consider in D_r the Poincaré metric and in M the metric δ. This implies that $\mathcal{U}(\mathcal{F}) \subset C^0(\mathbb{D}, M)$ is normal. □

Now, Theorem 1 will be a consequence of Proposition 2 and of the following result:

Theorem 2. *Let $\mathcal{F}$ be a foliation by curves on a compact complex manifold M. Suppose that $T_\mathcal{F}$ has a metric of negative curvature and that all singularities of $\mathcal{F}$ are non degenerated. Then there exists a continuous, complete and $\mathcal{F}$-ultrahyperbolic hermitian metric μ on $M \setminus sing(\mathcal{F})$.*

Proof. We prove first that, with a metric of negative curvature on $T_\mathcal{F}$, we can produce a hermitian metric on $M \setminus sing(\mathcal{F})$ which induces in the leaves of $\mathcal{F}$ a conformal metric with gaussian negative curvature (in general not bounded from zero). Let h be a fixed C^∞ hermitian metric on M and ν be a C^∞ metric of negative curvature on $T_\mathcal{F}$. Fix a covering $(U_j)_{j\in J}$ by Stein open sets of M and colections $(X_j)_{j\in J}$ and $(f_{ij})_{U_i\cap U_j\neq\emptyset}$, where

(i) X_j is a holomorphic vector field on U_j which defines $\mathcal{F}|_{U_j}$.

(ii) If $U_i \cap U_j \neq \emptyset$ then $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ and $X_j = f_{ij} \cdot X_i$ on $U_i \cap U_j$.

The bundle $T_\mathcal{F}$ can be defined as

$$T_\mathcal{F} = \uplus_{j\in J}(U_j \times \mathbb{C})/ \simeq$$

where $\uplus$ denotes disjoint union and $\simeq$ is the equivalence relation on $\uplus_{j\in J}(U_j \times \mathbb{C})$ defined by

$$(p, v_i) \simeq (q, v_j) \iff \begin{array}{l}(p, v_i) \in U_i \times \mathbb{C}\ ,\ (q, v_j) \in U_j \times \mathbb{C}\ ,\ p = q \\ \text{and } v_i = f_{ij}(p) \cdot v_j\ .\end{array} \tag{3}$$

For each $j \in J$, let us define $f_j\colon U_j \to (0, +\infty)$ by $f_j(p) = \nu(p, 1_j)$, where 1_j denotes the section of $T_{\mathcal{F}}$ which in the trivialization $U_j \times \mathbb{C}$ is given by $1_j(p) \equiv 1$. It follows from (3) that if $p \in U_i \cap U_j$, then $(p, 1) \in U_j \times \mathbb{C}$ is equivalent to $(p, f_{ij}(p)) \in U_i \times \mathbb{C}$, so that $1_j = f_{ij} \cdot 1_i$ on $U_i \cap U_j$. This implies that:

$$\begin{aligned} f_j(p) &= \nu(p, 1_j(p)) = \nu(p, f_{ij}(p) \cdot 1_i(p)) \\ &= |f_{ij}(p)|^2 \cdot f_i(p) \implies f_j = |f_{ij}|^2 \cdot f_i \text{ on } U_i \cap U_j\,. \end{aligned} \tag{4}$$

Now, consider the hermitian metric g_j on $U_j \setminus sing(\mathcal{F})$ defined by

$$g_j = f_j \cdot \frac{h}{h(X_j)}\,.$$

Since $X_j = f_{ij} \cdot X_i$ on $U_i \cap U_j \neq \emptyset$, we have

$$g_j = |f_{ij}|^2 \cdot f_i \cdot \frac{h}{h(f_{ij} \cdot X_i)} = f_i \cdot \frac{h}{h(X_i)} = g_i\,,$$

on $U_i \cap U_j \setminus sing(\mathcal{F})$. This implies that there exists a hermitian metric g on $M \setminus sing(\mathcal{F})$ such that $g|_{U_j \setminus sing(\mathcal{F})} = g_j$.

We now use that ν has negative curvature. This is equivalent to the fact that $\ln(f_j)$ is strictly pluri subharmonic (briefly spsh) for all $j \in J$ (cf. [G-H]). Let $p \in U_j \setminus sing(\mathcal{F})$, L be the leaf of $\mathcal{F}$ through p and let us prove that the conformal metric $g|_L$, induced by g on L, has gaussian negative curvature at p. Let $z\colon D_\epsilon \to L$ be the solution of the ordinary differential equation

$$\frac{dz}{dT} = X_j(z(T))$$

with initial condition $z(0) = p$, where $D_\epsilon = \{T \in \mathbb{C};\ |T| < \epsilon\}$, $\epsilon > 0$ is small. Then

$$\begin{aligned} z^*(g) &= g_{z(T)}(z'(T))|dT|^2 = f_j(z(T))\frac{h_{z(T)}(z'(T)))}{h_{z(T)}(X_j(z(T)))} \\ &= f_j(z(T))|dT|^2 := f(T)|dT|^2\,. \end{aligned}$$

On the other hand, the gaussian curvature of $g|_L$ at p is

$$k_g(p) = -2\frac{1}{f(0)}\frac{\partial^2 \ln(f)}{\partial T \partial \overline{T}}(0)\,. \tag{5}$$

Since $f(T) = f_j(z(T))$ and $\ln(f_j)$ is spsh, we get $k_g(p) < 0$. We observe that (5) implies that

$$k_g = -2\frac{1}{f_j^3}[f_j \cdot \mathcal{L}(f_j)(X_j) - |\partial f_j(X_j)|^2], \tag{6}$$

where $\mathcal{L}(f_j)$ is the Levi form of f_j, which is defined in a holomorphic coordinate system $(w_1, ..., w_n)$ by

$$\mathcal{L}(f_j) = \sum_{r,s=1}^{n} \frac{\partial^2 f_j}{\partial w_r \partial \overline{w_s}} dw_r d\overline{w_s}.$$

This implies, in particular, that k_g is of class C^∞. Therefore, if $sing(\mathcal{F}) = \emptyset$, then $k_g \leq -a^2 < 0$, for some $a > 0$, and we could apply Proposition 2 to this metric to obtain Theorem 1. However, if $sing(\mathcal{F}) \neq \emptyset$, we could have $\limsup_{q \to p} k_g(q) = 0$, where p is some singularity of $\mathcal{F}$. Our work now, will be to modify this metric in a neighborhood of the singularities of $\mathcal{F}$ and obtain another which is $\mathcal{F}$-ultrahyperbolic and complete. We need a definition.

Definition 3. Let U be a neighborhood of a point p in a complex manifold and g be a continuous hermitian metric on $U \setminus \{p\}$. We say that g is *complete at* p, if for any C^1 path $\gamma\colon [0, 1] \to U$, such that $\gamma(1) = p$ and $\gamma[0, 1) \subset U \setminus \{p\}$, we have $\ell_g(\gamma) = +\infty$, where

$$\ell_g(\gamma) = \int_{\gamma|_{[0,1)}} \sqrt{g}\,.$$

Remark 4. Let F be a finite subset of M. A continuous hermitian metric μ on $M \setminus F$ is complete if, and only if, it is complete at all points of F.

Lemma 1. *Let X be a holomorphic vector field on $B_r := \{z \in \mathbb{C}^n;\ |z| < r\}$, where $|z|^2 := \sum_{j=1}^n |z_j|^2$ and $r \leq 1$. Suppose that $0 \in B_r$ is the unique singularity of X in B_r, which is non degenerated. Then, for any C^∞ hermitian metric h on B_r and any pluri-harmonic function u on B_r, the hermitian metric*

$$\mu = \frac{\exp(u)}{\ln^2(|z|)} \cdot \frac{h}{h(X)}$$

satisfies the following properties:

(a) *μ is complete at $0 \in B_1$.*

(b) $k_\mu < 0$ *on* B_r *and* $\limsup_{q\to 0} k_\mu(q) < 0$.

Proof. Let $z(T)$ be the solution of $dz/dT = X(z(T))$ such that $z(0) = p$ and $f(T) = \frac{\exp(u(z(T)))}{\ln^2(|z(T)|)}$. It follows from (5) that

$$-\frac{1}{2}k_\mu(p) = \frac{1}{f(0)}\frac{\partial^2}{\partial T\partial \overline{T}}[u\circ z + \ln(\ln^{-2}(|z|))](0)$$
$$= \frac{\ln^2(|p|)}{\exp(u(p))}\frac{\partial^2 \ln(\ln^{-2}(|z|))}{\partial T\partial\overline{T}}(0)\ ,$$

because u is pluri-harmonic. On the other hand, if we denote by $<,>$ the hermitian product $< z, w > = \sum_j z_j\overline{w_j}$, then

$$\frac{\partial^2 \ln(\ln^{-2}(|z|))}{\partial T\partial\overline{T}} = \frac{1}{|z|^4|\ln(|z|)|}\left(|z|^2|X(z)|^2 - |< X(z), z >|^2\right) + \frac{|< X(z), z >|^2}{2|z|^4\ln^2(|z|)}\ , \tag{7}$$

as the reader can check, by using that $dz/dT = X(z)$. Let us denote by $\phi(z)$ the function in the right side of (7). Since $|< X(z), z >|^2 \leq |z|^2|X(z)|^2$ and $|< X(z), z >|^2 = |z|^2|X(z)|^2$ if, and only if $X(z) = \lambda\cdot z$, we get $\phi > 0$ which implies that $k_\mu < 0$ on $B_r\setminus\{0\}$. Now, we have

$$-\frac{1}{2}k_\mu(z) = \frac{\ln^2(|z|)}{\exp(u(z))}\cdot\phi(z) = \frac{|\ln(|z|)|}{\exp(u(z))}\left(\frac{|X(z)|^2}{|z|^2} - \frac{|< X(z), z >|^2}{|z|^4}\right) + \frac{|< X(z), z >|^2}{2|z|^4\exp(u(z))}\ .$$

If $|z| < e^{-1/2}$ then $|\ln(|z|)| > 1/2$ and so

$$-k_\mu(z) \geq \frac{|X(z)|^2}{\exp(u(z))|z|^2}\ ,$$

for $|z| < e^{-1/2}$. Since 0 is a non degenerated singularity of X, there exists $0 < c < 1$ such that $c^{-1}|z| \geq |X(z)| \geq c\cdot|z|$, for $|z|$ small, which implies that $\limsup_{q\to 0} k_\mu(q) \leq -a < 0$, where $a = c^2/\exp(u(0))$. This proves **(b)**.

Let us prove **(a)**. Since h is a continuous hermitian metric, if we fix $r_1 < r$, there exists $C > 1$ such that $C^{-1}|v|^2 \leq h_z(v) \leq C|v|^2$ and $\exp(u(z)) > C^{-1}$ for any $|z| < r_1$ and any $v \in \mathbb{C}^n$. It follows that

$$g_z(v) = \frac{\exp(u(z))h_z(v)}{h_z(X(z))\ln^2(|z|)} \geq C^{-3}\frac{|v|^2}{|X(z)|^2\ln^2(|z|)} \geq b\frac{|v|^2}{|z|^2\ln^2(|z|)}\ ,$$

where $b = c^2 \cdot C^{-3} > 0$. Since the metric $|dz|^2/|z|^2 \ln^2(|z|)$ is complete at $0 \in B_r$, μ is also complete at 0. □

Lemma 2. *Let $sing(\mathcal{F}) = \{p_1, ..., p_k\}$, h be a C^∞ hermitian metric on M and g be the hermitian metric defined before Lemma 1 from h and from the metric ν. There exists a continuous hermitian metric μ on $M \setminus sing(\mathcal{F})$, neighborhoods $W_j \subset V_j \subset U_j$ of p_j and holomorphic coordinate systems $\phi_j : U_j \to \mathbb{C}^n$, $j = 1, ..., k$, with the following properties:*

(a) *$U_i \cap U_j = \emptyset$ if $i \neq j$ and μ coincides with g on $M \setminus \cup_j V_j$.*

(b) *$\phi(p_j) = 0 \in \mathbb{C}^n$ and $\phi_j(W_j) = B_{r_j} = \{z \in \mathbb{C}^n;\ |z| < r_j\}$, where $r_j < 1$.*

(c) *$\mu = \phi_j^*(\frac{\exp(u_j)}{\ln^2(|z|)}) \cdot \frac{h}{h(X_j)}$ on W_j, where u_j is pluri-harmonic on B_{r_j} and X_j is a holomorphic vector field which defines $\mathcal{F}|_{U_j}$.*

(d) *μ is $\mathcal{F}$-ultrahyperbolic and complete on $M \setminus sing(\mathcal{F})$.*

Proof. First observe that **(c)**, Lemma 1 and Remark 4 imply that μ is complete on $M \setminus sing(\mathcal{F})$. Let us show how to modify the metric g in a neighborhood of p_j to obtain a metric satisfying property **(c)**. Fix holomorphic coordinate systems $\psi_j : U_j \to \mathbb{C}^n$, such that $p_j \in U_j$, $\psi_j(p_j) = 0 \in \mathbb{C}^n$, $\psi_j(U_j) = B_1$, $j = 1, ..., k$, and $U_i \cap U_j = \emptyset$ if $i \neq j$. Let X_j be a holomorphic vector field which defines $\mathcal{F}|_{U_j}$, so that, on U_j we have $g = f_j \cdot \frac{h}{h(X_j)}$. As we have seen the function $f := \ln(f_j)$ is C^∞ and spsh. Therefore, in the coordinate system $(U_j, z = \psi_j)$, we can write

$$f(z) = \ln(f_j)(z) = v_j(z) + \mathcal{L}_j(z) + R_j(z)\ ,$$

where, $\mathcal{L}_j$ is the positive definite Levi form

$$\mathcal{L}_j(z) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \overline{z_j}}(0) z_i \cdot \overline{z_j}\ ,$$

$v_j(z) = f(0) + k(z) + \overline{k(z)}$, where k is degree two complex polynomial

$$k(z) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(0) z_j + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial z_j}(0) z_i \cdot z_j$$

and

$$\lim_{z\to 0}\frac{R_j(z)}{|z|^2}=0\,.$$

Observe that v_j is pluri-harmonic.

Since $\mathcal{L}_j$ is positive definite, after a linear change of coordinates in $\mathbb{C}^n$, we can suppose that

$$\mathcal{L}_j(z)=\sum_{j=1}^{n}2|z_j|^2=2|z|^2$$

and that $\ln(f_j)(z)=v_j(z)+2|z|^2+R_j(z)$ is defined in $B_r\subset\psi_j(U_j), 0<r<1$. Let $\ell_\epsilon(z)=\epsilon+v_j(z)+|z|^2$. We assert that there exist $0<r_1<r_2<r_3<r$ and $\epsilon>0$ such that

$$\ln(f_j)(z)>\ell_e(z)\text{ if } r_2<|z|<r_3 \text{ and } \ln(f_j)(z)<\ell_e(z)\text{ if } |z|<r_1\,. \tag{8}$$

In fact, since $\lim_{z\to 0}|R(z)|/|z|^2=0$, let $0<r_3<r$ be such that $R(z)<\frac{1}{2}|z|^2$ for $|z|\le r_3$. If $\delta(z)=\ln(f_j)(z)-\ell_\epsilon(z)=|z|^2+R(z)-\epsilon$ and $|z|\le r_3$, we have

$$\frac{1}{2}|z|^2-\epsilon\le|z|^2-|R(z)|-\epsilon\le\delta(z)\le|z|^2+|R(z)|-\epsilon\le\frac{3}{2}|z|^2-\epsilon\,.$$

If we take $r_2=r_3/2$ and $\epsilon<r_3^2/8$, we get for $r_2<|z|<r_3$ that $\delta(z)\ge\frac{1}{2}|z|^2-\epsilon\ge r_3^2/8-\epsilon>0$, and so $\ln(f_j)(z)>\ell_e(z)$ for $r_2<|z|<r_3$. On the other hand, if $r_1<\sqrt{2\epsilon/3}<r_3/(2\sqrt{3})<r_2$, we get for $|z|<r_1$ that

$$\delta(z)\le\frac{3}{2}|z|^2-\epsilon<\frac{3}{2}r_1^2-\epsilon<0\,,$$

and so $\ln(f_j)(z)<\ell_\epsilon(z)$ if $|z|<r_1$, which proves the assertion. Define $k_j\colon B_r\to(0,+\infty)$ by

$$\begin{cases} k_j(z)=\exp(\ell_\epsilon(z)) & \text{if } |z|<r_1\\ k_j(z)=\max\{\exp(\ell_\epsilon(z)),f_j(z)\} & \text{if } r_1\le|z|\le r_2\\ k_j(z)=f_j(z) & \text{if } |z|>r_2\end{cases} \tag{9}$$

It follows from (8) that k_j is continuous and $\ln(k_j)$ is spsh (because $\ln(f_j)$ and ℓ_ϵ are spsh). We do this construction for all singularities $p_1,\ldots,p_k\in sing(\mathcal{F})$ and define a metric μ_1 on $M\setminus sing(\mathcal{F})$ by

$$\begin{cases}\mu_1=k_j\circ\psi_j\frac{h}{h(X_j)} & \text{on } \cup_{j=1}^k\psi_j^{-1}(B_{r_3})\\ \mu_1=g & \text{on } M\setminus\cup_{j=1}^k\psi_j^{-1}(B_{r_3})\end{cases} \tag{10}$$

This metric is clearly continuous and satisfies the following property: it is $\mathcal{F}$-ultrahyperbolic in any open set of the form $M \setminus A$, where A is an open neighborhood of $sing(\mathcal{F})$. This last assertion follows from Proposition 1. We leave the details of its proof for the reader.

Now, we will modify the metric μ_1 in $\psi_j^{-1}(B_{r_1})$, $j = 1, ..., k$, in order to obtain the metric μ of Lemma 2.

Assertion. *Let* $\varphi(t) = (\ln(t))^{-2}$ *and* $\varphi_{a,\alpha}(t) = a\varphi(t_o(\frac{t}{t_o})^\alpha) = a\varphi(t_o^{1-\alpha}t^\alpha) = a\varphi([t/t_o^{(\alpha-1)/\alpha}]^\alpha)$. *Given* $0 < t_o < 1$ *there are* $\epsilon, a > 0$ *and* $0 < \alpha < 1$ *such that*

(i) $\varphi_{a,\alpha}(t) \le e^t$ *if* $t_o \le t \le t_o + \epsilon$.

(ii) $\varphi_{a,\alpha}(t) \ge e^t$ *if* $t_o - \epsilon \le t \le t_o$.

Proof. Let $a > 0$ be such that $a\varphi(t_o) = e^{t_o}$. Observe that $\varphi_{a,\alpha}(t_o) = a\varphi(t_o) = e^{t_o}$. On the other hand,

$$\varphi'_{a,\alpha}(t_o) = a\alpha\varphi'(t_o) = \frac{2a\alpha}{t_o|\ln^3(t_o)|} = \frac{2\alpha e^{t_o}}{t_o|\ln(t_o)|} > 0\,.$$

Therefore, there exists $\alpha > 0$ such that $0 < \varphi'_{a,\alpha}(t_o) < e^{t_o} = \frac{de^t}{dt}|_{t=t_o}$. On the other hand,

$$\varphi'_{a,\alpha}(t_o) = \frac{2\alpha e^{t_o}}{t_o|\ln(t_o)|} < e^{t_o} \implies \alpha < \frac{t_o|\ln(t_o)|}{2} \le \frac{1}{2e} < 1\,,$$

because $t_o < 1$, as the reader can check. This implies the assertion. □

Observe that the function $\varphi_{a,\alpha}$ of the assertion is defined for $t < 1$ because $t_o^{(\alpha-1)/\alpha} > 1$.

Let k_j and $r_1 < r_2$ be as in (9), so that $k_j(z) = \ell_\epsilon(z) = \epsilon + v_j(z) + |z|^2$ if $|z| < r_1$. Fix $0 < r_o < \min\{1, r_1\}$ and consider a function $\varphi_{a,\alpha}(t)$ defined for $t < 1$, as in the assertion, by

$$\varphi_{a,\alpha}(t) = a\varphi([t/r_o^{2(\alpha-1)/\alpha}]^\alpha)$$

where $\varphi_{a,\alpha}(r_o^2) = e^{r_o^2}$, $\varphi_{a,\alpha}(t) \le e^t$ if $r_o^2 \le t \le r_o^2 + \epsilon$ and $\varphi_{a,\alpha}(t) \ge e^t$ if $r_o^2 - \epsilon \le t \le r_o^2$. Define m_j by

$$\begin{cases} m_j(z) = \exp(|z|^2) & \text{for } r_o \le |z| < r_1 \\ m_j(z) = \varphi_{a,\alpha}(|z|^2) & \text{for } |z| \le r_o \end{cases}$$

Since $\varphi_{a,\alpha}(r_o^2) = e^{r_o^2}$, m_j is well defined and continuous. Moreover,

$$m_j(z) = \max\{\exp(|z|^2), \varphi_{a,\alpha}(|z|^2)\} \text{ if } r_o^2 - \epsilon < |z|^2 < r_o^2 + \epsilon \,. \tag{11}$$

Define $n_j : B_{r_3} \to (0, +\infty)$ by

$$\begin{cases} n_j(z) = k_j(z) & \text{if } r_1 < |z| < r_3 \\ n_j(z) = \exp(\epsilon + v_j(z)) \cdot m_j(z) & \text{if } |z| \le r_1 \end{cases} \tag{12}$$

It follows from (9) that n_j is continuous. Do this construction for all $j = 1, \ldots, k$ and define the metric μ on $M \setminus sing(\mathcal{F})$ by

$$\begin{cases} \mu = n_j \circ \psi_j \frac{h}{h(X_j)} & \text{on } \cup_{j=1}^k \psi_j^{-1}(B_{r_3}) \\ \mu = \mu_1 & \text{on } M \setminus \cup_{j=1}^k \psi_j^{-1}(B_{r_3}) \end{cases}$$

It follows from (10) and (12) that μ is continuous. On the other hand, if we consider the change of variables $w = z/r_o^{\frac{\alpha-1}{a}}$, $z \in B_{r_1}$, we have for $|w|$ small enough, that

$$n_j(w) = \exp(\epsilon + v_j(r_o^{\frac{\alpha-1}{a}} w))\varphi_{a,\alpha}(r_o^{\frac{2(\alpha-1)}{a}} |w|^2) = \exp(u_j(w)) \ln^{-2}(|w|) \ ,$$

where $u_j(w) = \ln(a/4\alpha^2)(\epsilon + v_j(r_o^{\frac{\alpha-1}{a}} w))$ is pluri-harmonic. Now, Proposition 1, (11) and Lemma 1 imply that μ is $\mathcal{F}$-ultrahyperbolic and complete. This proves Lemma 2 and Theorem 2. □

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